Electromagnetism

01/10/98

Vector Algebra

(del=upsidedown triangle!)

Div,Grad&Curl

Div-"Eminating field lines."
Divergence

del.D=(i.Ð/Ðx+j.Ð/Ðy+k.Ð/Ðz).D
D=ai+bj+ck
del.D=(Ða/Ðx+Ðb/Ðy+Ðc/Ðz)
eq 1

Divergence is a scalars quantity

Grad

del.Q=(i.Ð/Ðx+j.Ð/Ðy+k.Ð/Ðz).Q
eq2

For electric fields

E=-*d*V/*d*x

Therefore,

E=-Grad(V(x,y,z))

Curl

del^H

Example, field lines around a straight current line are circular, they curl!

del^H=I/A amps

eq 3

Guass's theorem

fig 1

"Total normal flux field of electric field through a closed surface is equal to total enclosed charge, divided by permittivity of free space"

eq 4

(AmpŠres law and electromagnetic induction for tomorrow.)

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AmpŠres Law

fig 1

Electromagnetic Induction

Faraday's Law

Gives the magnitude of the e.m.f.. It states that,

Induced e.m.f. is proportional to change of flux
e.m.f. proportional Ð( )/Ðt

Lenz's Law

This gives the direction of the induced e.m.f.. The direction is such to oppose the change in flux

e.m.f.=-Ð( )/Ðt
eq 1

Vector Differential Form of the above equations

The electric displacement, D, is equal to,

D=*E*.E
eq 2
Gauss's Law: divD=ç=*del*.D

eq 3
AmpŠres Law: curH=J=del H

eq 4
Lenz and Faraday's Law: CurE=-ÐB/Ðt=del E

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Modifacation to AmpŠres Law (Maxwell)

Note, to date what has be en considered as "conventional current" is actually the "Conduction Current".

When considering a current through a circuit which has a capacitance, we need to develop some continuity of current to account the lack of current past the capacitor.
This arises because AmpŠres law is inadequate. It only works with conduction current in it's current form. How do we maintain continuity of current ? Is there a different current between the plates of the capacity ?

AmpŠres law breaks down when

Integ (over $) {J.dS} =0

i.e. when the surface integration cuts the region between the plates.

del.(deH)=del.J
=0
del.J=0

A way around this is to have a hypothetical charge density, ç, between the plates.
Therefore , **del.J** is no longer zero (according to AmpŠres law in it's present form).

del.J=-Ðç/Ðt
del.D=ç
Therefore, del.J+ÐD/Ðt)=0
del.J=-Ð(del.D)/Ðt

Where **D** is the "Displacement current".

See Example 1

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Summary

Maxwell's (correction of AmperŠ's law)

del H=J+ÐD/Ðt

del E=-ÐB/Ðt
del.B=0
(B=æ.H)

del.D=ç
J=àE
(Ohm's law)

Prediction of electromagnetic waves from Maxwell's equations

del H = àE + *E*.(ÐE/Ðt)
(D=*E*E)

del E = -æ.(ÐH/Ðt)
(B=æH)

del delE=-æ.(Ð/Ðt{delH})

=-æ.Ð/Ðt{àE + *E*.(ÐE/Ðt)}
=-æà.(ÐE/Ðt)-æ*E*.(ÐýE/Ðtý) eq(1)
=del.(delE)-delýE

del.E=0, i.e no free charge, i.e. charge free region.

i.e. no field generated within the medium to which the equation is applied.

Therefore,

del delE=-delýE eq(2)

Combine (1) & (2)

-delýE=-æà.(ÐE/Ðt)-æ*E*.(ÐýE/Ðtý)

Therefore,

delýE-æà.(ÐE/Ðt)-æ*E*.(ÐýE/Ðtý)=0
or
delýH-æà.(ÐH/Ðt)-æ*E*.(ÐýH/Ðtý)=0

In general where **u** is **H** or **B**,

delýu-æà.(Ðu/Ðt)-æ*E*.(Ðýu/Ðtý)=0

Consider the possible solution (in a single dimension),

u=u$_o$.exp{-*α*x}.exp{j(*ω*t-á.x)}
(single dimension equation)
(fig 1)

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The above equation is in a single dimension. It is a plane wave propagating in the x direction. á is the wave number and *α* is the attenuation coefficient. It can be rewritten as,

u=u.exp{j*w*t-*y*x}

Where *y*=*a*+já

We now need to find the range of differentials of **u**.

Ðu/Ðx=Ð/Ðt{u.exp(j*w*t-*y*x)}
=Ð/Ðt{u.exp{j*w*t}.exp{-*y*x}}
=u.exp{j*w*t}.(-*y*.exp{*y*x})-*y*.uexp{j*w*t-*y*x}

Therefore,
Ðýu/Ðtý=*yý*u.exp{j*w*t-*y*x}

Substituting into the equation for **u**, we get,

***yý*u.exp{j*w*t-*y*x}-•.æ.j.*w*.uexp{j*w*t-*y*x}+æ*E.qwý*.uexp{j*w*t-*y*x}=0**

***yý*-•.æ.j.*w*+æ*E.wý*=0**

***yý*=(•.æ.j.*w*-æ*E.wý*)**

But,

***y*=*a*+já**
***yý*=*aý*-áý+2j*a*á**
aý*-áý+2j*a*á=•.æ.j.*w*-æ*E.wý

Real: *aý*-áý=-æ*E.wý*
Imag: 2*a*á=•.æ.*w*

We now need to solve for *a* and á. Taking a simple case in a non-conducting medium, i.e. •=0. If á is zero, we do not get a progressing wave, just something that oscillates in time, it doesn't go anywhere. So *a* must have to be equal to zero. So, in a non-conducting medium, there is no attenuation.

2*a*á=0 if *a*=0
then, *aý*-áý -> -áý
Therefore, -áý=-æ*E.wý*
Therefore, á, wave number,*w*=.(æ*E*)

Hence the velocity of the wave*w*/á
v = 1/(æ*E*)

w=2pi/f, á=2pi/lambda

<u>Speed of light</u>

So in free space, v = 1/(æ$_o$*E*$_o$)$^{«}$. æ$_o$=4pi 10^{-7} H/m and *E*$_o$=1/36pi 10^{-9} F/m

With these numerical values, v=3 10^{8} msîû. This is the velocity of light in a vacuum.

<u>Refractive index</u>

Refractive index=vel in free space/vel in medium (non-cond.)

=(1/(æ$_o$*E*$_o$)$^{«}$)/(1/(æ*E*))
=(*E*æ/*E*$_o$æ$_o$)$^{«}$
for a non-magnetic medium æ=æ$_o$
=(*E*/*E*$_o$)$^{«}$
=*K*$^{«}$

K is the <u>dielectric constant</u> (the relative permativitty). Hence refractive indices from measurement of the dielectric property.

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<u>Nature of the wave</u>

fig 1

(Shows a TEM, "Transverse EM" wave)

The following is a proof that an electromagnetic wave is transverse wave.

<u>E</u>$_x$ $_y$=vector in direction of propagation

Consider,

E=E$_o$.exp(j(wt-áx))

ie assume wave travelling in z direction E is a function of Z and t.

<u>viz</u>

ÐE/Ðx=ÐE/Ðy=0

because E is not a function of x or y.

del.<u>E</u>=0 (ç=0)

and there are no free charges generating a field.

del.<u>E</u>=(ÐE$_x$/Ðx)+(ÐE$_y$/Ðy)+(ÐE$_z$/Ðz)

Therefore,
del.E=ÐE$_z$/Ðz=0

Therefore, E_z cannot be a function of z. Therefore, there is no component of electric field in the direction of propagation.

If there is no component of E along the Z axis, then E_x or E_y are non zero. Therefore, choose the non-zero component of E to lie along the z axis (E_x<>0, E_y=0, plane polarised).

Having chosen the electric vector to lie along the x axis.

E=i.E$_x$(z,t)

del E=-ÐB/Ðt
(see eq 1)

ÐB$_x$/Ðt=ÐH$_z$/Ðt=0 (1)
ÐE$_x$/Ðz=-ÐB$_y$/Ðt (2)
-ÐE$_x$/Ðy=-ÐB$_z$/Ðt (3)

By (3),

-ÐE$_x$/Ðy=-ÐB$_z$/Ðt=0

E_x is a function of z and t only. Therefore, B_z=H_z=0. Therefore no component of B or H along the direction of propagation.

Waves in conductive medium

In such cases the attenuation of the wave is non zero. •<>0, hence we write,

delý.u-æ•(Ðu/Ðt)-æE(Ðýu/Ðtý)=0
u=u$_o$.e^(-*a*z)).e^(j(*w*t-áz))
=u$_o$e^(i*w*t-*y*z)
(*y*=*a*+já)

***y*ý=æ•j*w*-æ*Ew*ý**
***a*ý-áý=-æ*Ew*ý**

:Real 2*a*á=æ•*w*
:Imag 4*a*ýáý=(æ•*w*)ý
áý=(æ•*w*)ý/4*a*ý

Substitute and hence,

4*a*^{4}+4æ*Ew*ý*a*ý-æý•ý*w*ý=0

***a*=*w*.((æ*E*/2).ñ(1+(•ý/*E*ý*w*ý)-1)**

We have to take the positive root, or *a* would be negative and energy conservation would be violated.

_a_=_w_.((æ_E_/2).(1+(•ý/_E_ý_w_ý)-1)

For á we have (from a similar derivation),

á=_w_.((æ_E_/2).ñ(1+(•ý/_E_ý_w_ý)+1)

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For a good conductor the limiting factor is if • is large. The determining factor for *a* and á is,

• **/_Ew_**

Then if •¯*Ew*, then the respective surd term in *a* and á approximates to,

• **/_Ew_**

Therefore in the limit **/_Ew_** being large, we get the relationship,

_a_=á=(æ•_w_/2)

Now let,

=•/_Ew_

(note, this is <u>not</u> magnetic flux!)
So, for a good conductor is small. If <=1/50 then the approximation above is valid to between ñ1%. <u>Then</u> the material is classed as a good conductor.

e.g.

Copper,
•=10^7 1/ohm.m
E=*E*ₒ=1/36pi * 10^-9 F/m
æ=æₒ=4pi*10^-7 H/m
Conductivity of Cu breaks down at 1.421E+17 Hz

The ratio of does have a physical meaning, it is shown as follows,

=
=displacmenet current density/conduction current density
=(ÐD/Ðt)/•E
(J=•E)
=_Ew_E/•E
=_Ew_/•

In a conducting media ÐD/Ðt is small, which leads to be small as stated above.

Problem

Phase velocity of wave,
a)for a non-conducting medium,
b)for a conducting medium

Phase velocity=
a)v=w/á,
b)v=(w.2)/(æ.•.w)=(2w/æ•)

We can see here that in a non-conducting medium the velocity is frequency indenpendant. It is then frequency dependant in a conducting medium, this phenonema is called **dispersion**, the different frequency components travel at different speeds in the medium.

Energy Flow : Poyntings vector

Consider,

del.(E)
=H.del E-E.del H
=H.(-ÐB/Ðt)-E.(J+ÐD/Ðt)
=-æH.(ÐH/Ðt)-EE.(ÐE/Ðt)-E.J
=-Ð/Ðt {«æHý-«EEý}-E.J
(obtained from chain rule)

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Now take integrals of both sides,

Integ_vol {} (del.(E))=Integ_vol {} (-Ð/Ðt {«æHý-EEý}.dT) - Integ_vol {} ((E.J).dT)

Using Stoke' theorem,

Integ_vol {} (del.(E).dT) = Integ_surface {} (E H).dS

Therefore,

Integ_surface {} (E).dS = Integ_vol {} (-Ð/Ðt {«æHý-EEý}.dT) - Integ_vol {} ((E.J).dT)

Where **E** is the rate of change of energy in the volume.
On the right hand side, the first term is for dielectric materials absorbing energy and the second term is for conduction in the material (if possible).

We now state,

$$\underline{\mathbf{P}}=\underline{\mathbf{E}}$$

Where **P** is the "Poynting Vector".

Average Power in a Wave

$$\mathbf{E}=\mathbf{E}_o.\cos(\omega t)$$
and
$$\mathbf{H}=\mathbf{H}_o.\cos(\omega t+\ )$$

  is the phase difference between **E** and **H**.

The Poynting vector is,

$$\mathbf{P}=\mathbf{E}\ \mathbf{H}$$
$$=\mathbf{E}_o\mathbf{H}_o.[\cos(\omega t+\ )+\cos(\ )]$$
$$\text{Avg. Pow.} = «.\mathbf{E}_o\mathbf{H}_o.\cos(\ )$$

For a non-conducting medium =0 so cos( )=1 and the average power is a maximum. This is analogous to an alternating current through a resistor, with no inductance.

For a conducting medium ...

Intrinsic Impedance (for a medium)

Is defined as,

$$n=\mathbf{E}_x/\mathbf{H}_y$$

$$\text{del } \underline{\mathbf{E}}=-\text{Đ}\underline{\mathbf{B}}/\text{Đ}t$$
(see eq3 for 01/10/98, matrix)

$$\text{Đ}\mathbf{E}_x/\text{Đ}z=-\text{Đ}\mathbf{B}_y/\text{Đ}t$$

$$u=u_o.e\text{^}(j\omega t-yz)$$

Therefore,

$$-y\mathbf{E}_x=-j\omega\mathbf{B}_y.$$

Hence,

$$n=\mathbf{E}_x/\mathbf{H}_y=j\omega æ/y$$

n=j*w*æ/(*a*+já)
=(j*w*æ/*a*ý+áý).(*a*-já)
=(*w*æ/(*a*ý+áý)).j(*a*-já)
=(*w*æ/(*a*ý+áý)).(á+j*a*)

For a non conductor *a*=0 (ie no attenuation), hence =0. Threrfore***n*** is real ! This means that **E/H** is real and **E** and **H** are in phase. For a good conductor *a* is approximatley equal to á, which in turn is equal to **(æ•*w*/2)**. If *a* and á are equal then the complex argument is =pi/4. So the phase difference between E and H is pi/4.

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The magnitude of the intrinsic impedance in an conducting medium

|*n*|=(*w*æ/(*a*ý+áý)).(áý+*a*ý)
=*w*æ/(*a*ý+áý)

For a good conductor,
***a*=á=((æ•*w*)/2)**
|*n*|=æ*w*/(æ•*w*)

|*n*|=(æ*w*/•)

sqr((4*pi*10**-7)*(2*pi*10**10)/5.8e7)=0.0368961345533

The magnitude of the intrinsic impedance in an non-conducting medium (perfect medium) (*a*=0)

|*n*|=æ*w*/á

á=*w*.(æ*E*)

Hence,
|*n*|=(æ/*E*)

sqr((4*pi*10**-7)/((1/(36*pi))*10**-9))=376.991118431

Attenuation of EM waves in a conducting medium. The "Skin Effect"

a is the rate of attenuation. For a good conductor,

$$\boldsymbol{a}=(æ•w/2)^{\frac{1}{2}}$$

We can rewrite our general form of H or E as,

$$\mathbf{u=u_o.exp\{-x/small\_delta\}.exp\{j(\omega t-áx)\}}$$

Clearly,

$$\mathbf{small\_delta=1/a}$$

Where small_delta is called the **Skin Depth**. Within a distance small_delta, the wave will decay by 1/e.
The magnitude of small_delta is,

$$\mathbf{small\_delta=(2/æ•w)^{\frac{1}{2}}}$$

(See skin effect sheet!)

fig 1

We write about conductors only conducting within the *skin depth* and *imperfect dielectrics* from now on.

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Ionised Gases : Ionosphere gases or Plasmas

These gases have a density of 10^{11} electrons / mü in the ionosphere and 10^{18} electrons / mü in plasmas.

When a T.E.M. interacts with an ionised gas, it is mainly the E field responsible for the observed behaviour. The force acting on the electrons by a T.E.M. can be expressed as,

$$\mathbf{\underline{F}=e\underline{E}+e\underline{v}\ æ\underline{H}}$$
$$\mathbf{=F_{elec}+F_{mag}}$$

$$\mathbf{|F_{mag}|/|F_{elec}|=evæH/eE} \quad \textbf{(approx)}$$
$$\mathbf{=væ(H/E)}$$
$$\mathbf{=væ.(E/æ_o)^{\frac{1}{2}}}$$

=v/c (approx)

Fmag/felec=10 @ 1MHz

Therefore $F_{mag} << F_{elec}$. Therefore, we can neglect the magnetic component of the wave.

We make several assumptions to make our model;

(1) H effect << E effect

(2) Assume medium's behaviour is mainly due to it's electrons.

(3) No collisions between electron and <u>either</u> positive ions or neutral particles. e.g. low pressure gases.

(4) No thermal motion.

In this model, the E field acting on the electrons is written as,

$$E=E_o.\exp\{j\omega t\}$$

This is at t=0, x=0 and x'=0. The equation of motion can be written as,

$$e\underline{E}=m.x''$$

$$eE_o.\exp\{j\omega t\}=m.x''$$

Note, there is no damping (dependant on x') and no resisting force (dependant on x).

Integrating both sides of the above equation,

$$eE_o.\exp\{j\omega t\}=m.x''$$

once,

$$eE_o(1/j\omega).\exp\{j\omega t\}+constant=m.x'$$

$$\Rightarrow x'=eE/j\omega m$$

and twice

$$-eE_o(1/\omega^2).\exp\{j\omega t\}=m.x$$

$$\Rightarrow x=-eE/\omega^2 m$$

Now consider the current density,

$$J=\bullet\underline{E}$$

$$=N.e.x'$$

$$=Ne(eE/j\omega m)$$

N=number of charge carriers, per unit volume.

Therefore,

•　　=Neý/j*w*m
=-jNeý/*w*m

•, the conductivity, is an imaginary number. Considering ohm's law, J=•E. If • is complex, then J and E are out of phase by pi/2 (looking on an Argan diagram, with J as real and E as imaginary!). It actually behaves like a pure inductance (by mathematical analogy).

Now consider,

del H=J+ÐD/Ðt

J is the current density of the electrons and the displacement current term represents the free space between the electrons. So we write,

del H=J+Ð(*E*E)/Ðt
=J+*E*ₒ.(ÐE/Ðt)
=•E+*E*ₒ.(ÐE/Ðt)
= -(jNeýE/*w*m) + *E*Ð/Ðt {E.exp{j*w*t}}
= -(jNeýE/*w*m) + *E*j*w*E

del H=j*w*{*E*-(Neý/*w*ým)}E

Since this has apparently has an imaginary conductance, we now try and consider it as a dielectric. So,

***E*={*E*ₒ-(Neý/*w*ým)}**

i.e. The effects of both the motion of the electrons and free space in which they are sitting is determined by a perfect dielectric medium. We now write,

del H=j*wE*E

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the j*wE* term is a description equivalent to dielectric behaviour.

The above comes from considering the material as a perfect dielectric where del H=ÐD/Ðt=*E*.ÐE/Ðt and E=E.exp{j*w*t}. Now if we factorise the equation above,

del H=E((-jNeý/*w*m)+j*wE*)

if the ionised gas is considered as a dielectric medium. Then,

j*wE*=-(jNeý/*w*m)+j*wE*ₒ

Therefore,

$$\boldsymbol{E=[E_o-(\text{Neý}/w\text{ým})]}$$

The (apparent) dielectric constant (relative permittivity) is then,

$$\boldsymbol{K=E/E_o}$$
$$\boldsymbol{=[1-(\text{Neý}/w\text{ým}E_o)]}$$

Assume a non-magnetic medium, hence æ=$æ_o$. Hence the phase velocity of the wave is,

$$\boldsymbol{c=1/(æE)}$$
$$\boldsymbol{=1/(æ_oE)^{«}}$$

The refractive index is then written as,

n=vel in free space/vel in medium

$$\boldsymbol{=(1/(æ_oE_o)^{«})/(1/(æ_oE)^{«})}$$
$$\boldsymbol{=K^{«}}$$

Therefore, the refractive index is,

$$\boldsymbol{n=[1-(\text{Neý}/Ew\text{ým})]}$$

We alternatively write,

$$\boldsymbol{n=[1-(w_p/w)\text{ý}]^{«}}$$

Where,

$$\boldsymbol{w_p=(\text{Neý}/E_o\text{m})^{«}}$$

This is refered to as the **Plasma frequency** (or less comonly used, "Langmuire Frequency").

The refractive index can become imaginary now though. Consider the following cases

a) $w>>w_p$
{1-(w_p/w)ý} is positive, so the refractive index is also real. *K* for the dielectric behaviour is also real. Also, $E=E_o$[1-(w_p/w)ý]. The wave number is written as,

$$\boldsymbol{á=á_o.n}$$
$$\boldsymbol{=(2pi/lambda).n}$$
$$\boldsymbol{=(2pi/lambda).\{1-(w_p/w)\}^{«}}$$

Also,

$$\boldsymbol{u=u_o.\exp\{-\alpha x\}.\exp\{i(wt-áx\}}$$

á is then real, attenuation is zero. Real wave number means the wave will still propagate.

b) $w << w_p$
á=(2pi/lambda).{1-(w_p/w)} . Now we write á=j|á|. Hence,

u=u_o.exp{j(wt-áx)}
=u_o.exp{j(wt-j|á|x)}
=u_o.exp{jwt}.exp{|áx|}

(Try N=1e11, e=1.67e-19, E_o=(1/36pi)e-9, m=9.31e-31)

c) $w = w_p$
Recall that,

J=•E=-(jNeý/wm).E

Which is inductive because, J lags E by pi/2. Also,

ÐD/Ðt=jwEE

This is capacitive because, ÐD/Ðt lags E by pi/2. Now we write,

J/(ÐD/Ðt)={-jNeýE/wm}/{jwEE}
=(Neý/$w$ýmE)
=(w_p/w)ý
=1

the two currents are equal in magnitude and out of phase by pi. In analogy to an LCR circuit we have **resonance** A consideration of resonant behaviour is beyond the scope of this model.

06/11/98

Waves at Boundires

Boundry conditions

The tangential component of electric field is continuous across a boundry. The Normal component of D field is countinuous across boundry.

Consider an upolarised beam incident on a mirror with reflected and refracted beams. The unpolarised incident beam is represented by two perpendicular polarisations one component is perpendicular to the plane of incidence the other is parallel.

fig one

(a) E perpendicular to plane of incidence

Plane of incidence (refracted and reflected in y=0). The boundary is in the plane z=0. We now equate the tangential componenets of E on each side of the boundary, atthe boundary.

$E_y+E''_y=E'_y$ (1)
incid+reflect=refract
lower medium=upper medium

For the H vector of the TEM wave is,

$H_x=-H\cos(i)$
$=-(E_y/n).\cos(i)$
n is the intrinsic impedance.

For the refracted wave,

$H'_x=-(E'_y/n')\cos(i)$
n' is the intrinsic impedance of the upper medium

For the reflected wave,

$H''_x=+(E''_y/n).\cos(i)$

Now equate the tangential components of the magnetic fields at the boundary,

$-(E_y/n).\cos(i)+(E''_y/n).\cos(i)=-(E'_y/n')\cos(i)$ (2)

We now need to eliminate the refracted (transmitted) component. Take equation (2),

$-(E_y-E''_y).(\cos(i)/n)=-(E'_y/n').\cos(r)$

Consider Snell's law,

$\sin(i)/\sin(r)=n'/n$

and recall the following,

$n'/n=(E'æ'/Eæ)$
$n=(æ/E)$
$n'=(æ'/E')$

Then we get from equation (2),

$E_y - E''_y = E'_y.(\tan(i)/\tan(r)).(æ/æ')$

multiply equation (1) by **(ætan(i)/æ'tan(r))** and then subttract equations (1) from (2) and assume $æ=æ'=æ_o$. Then we get,

$E''_y/E_y = -\sin(i-r)/\sin(i+r)$

<u>The First Fresnel Equation</u>

Consider an air to glass boundary. n<n' and i>r so sin(i-r) will always be positive and so will sin(i+r) (for 0<i<pi/2). This means that E''_y/E_y will be negative and a phase change (of pi) will occur.
For near normal incidence,

$E''_y/E_{y=(n'-n)/(n'+n)}$

The reflection coefficient (using intensities) is,

$(E''_y/E_y)ý=_{(n'-n)ý/(n'+n)ý}$

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See Fig.1

fig 1

From fig. 1,

$H_y=E/n$
$E_x=+E.\cos(i)$

$H'_y=E'/n'$
$E'_x=-E'.\cos(i)$

$H''_y=E''/n''$

$E''_x=-E''_x.\cos(i)$

Equate tangential components of the E and H fields, on each side of the boundry.

$E_x+E''_x=E'_x$
$(E-E'').\cos(i)=E'.\cos(r)$
$E-E''=E'.(\cos(r)/\cos(i))$ (1)

For tangential components of the H field

$H_y+H''_y=H'_y$

Therefore,

$E+E''/n=E'/n$

Therefore,

$E+E''=(n/n').E'$ (2)

$n/n'=(æE'/Eæ')^«$
$n'/n=(æ'E/E'æ)^«$

Therefore,

$n/n'=(n'/n).(æ/æ')$
$=(\sin(i)/\sin(r)).(æ/æ')$

(1) * n/n'
(2) * $\cos(r)/\cos(i)$
subtract eq (1) from eq (2)
then,

$(n/n').(E-E'')=E'.(\cos(r)/\cos(i)).(n/n')$
$(E+E'').(\cos(r)/\cos(i))=(n/n').(\cos(r)/\cos(i)).E'$

Therefore,

$(E+E'').(\cos(r)/\cos(i))=(E-E'').(\sin(i)/\sin(r))$

Therefore,

$E''.(\cos(r)\sin(r)+\cos(i)\sin(i))=E.(-\cos(r)\sin(r)+\sin(i)\cos(i))$

$E''/E=(\sin(i)\cos(i)-\cos(r)\sin(r))/(\cos(i)\sin(i)+\cos(r)\sin(r))$

$E''/E=\tan(i-r)/\tan(i+r)$

E is in the plane of incidence. The latter equation is known as the **2nd Fresnel equation**

Near normal incidence (i=0 deg), E''/E gives nearly 4%. Near grazing incidence (i=90 deg) E''/E is approxiamtley 1, i.e. nearly 100%.

If **i+r=pi/2**, then **tan(i+r)=infinity**, so our reflection coefficient goes to zero. The **Brewster angle** is the angle at which no relfection occurs.

If i+r=pi/2, then sin(r)=cos(i). Then by using Snell's law,

$\sin(i)/\sin(r)=n'/n$
$\tan(i_B)=n'/n$

For an air to glass boundary, the brewster angle, i_B, is 54 degrees.

Upon inspection of the second Fresnel equation, if the angle of incidence is geater than the Brewster angle, the numerator is negative and there is a subsequent phase change of pi. If the angle of incidence is less than the Brewster angle, then there is no phase change (numerator stays positive).

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If we now consider an TEM wave traveling from a dense to a less dense medium.

Reflection at dense to rare boundaries

One feature of such situations is total internal reflection, when $i=i_c$ i is the **critical angle** if $i>i_c$ then **total internal reflection** occurs.

fig 1

Let consider several cases now,

þ $i<i_c$

Polarisations parallel to the incidence plane, hits zero intensity at the brewster ange. Both parallel and perpendiculr polariziation goto 100% at the critical angle.

The reflections for the two polarised components follow the Fresnel eqaution, viz -sin(i-r)/sin(i+r) and tan(i-r)/tan(i+r) giving 100% reflection fr $i>i_c$

Phase on reflection: For E perpendicular to the plane of incidence,

E”/E=-sin(i-r)/sin(i+r)

This is positive and hence no phase change. Now, for E parallel to plane of incidence,

E”/E=tan(i-r)/tan(i+r)

This becomes negative, so we get a phase change of pi. Once i is greater than the Brewster angle tan(i+r) becomes negative too, so the ratio E”/E is positive. Therefore, there is no phase change on reflection.

$i>i_c$

The reflection should be 100%. Phase changes on reflection ?
Snell’s law (n’ for air and n for glass),

sin(i)/sin(r)=n’/n (n’<n) (1)

At the critical angle, i_c,

sin(i_c)=n’/n

If we increase i beyond i_c then sin(i) is greater than n’/n

sin(i)>n’/n

From eq(1),

sin(r)>1 (apparently!)

cos(r)=ñ(1-siný(r))^«
=ñj.((n’/n)ý.siný(i)-1)^«

The problem now, which root to choose, positive or negative. If we take the negative sign, otherwise the wave in the upper (rare) medium grows in amplitude (which violates conservation laws). Use this expression i the two Fresnel equations: (viz:)
Case 1, E perpendicular to the plane of incidence,

E”/E=-sin(i-r)/sin(i+r)
=-(sin(i)cos(r)+sin(r)cos(i))/(sin(i)cos(r)+cos(i)sin(r))
=-(cos(r).(n’/n)+cos(i))/((n’/n).cos(r)+cos(i))
E”/E=((j(siný(i)-(n’/n)ý)^«)+cos(i))/((-j(siný(i)-(n’/n)ý)^«)+cos(i))

|E”/E|=1

This means we have 100% percent reflection for i>i_c. We can now obtain the phase angle thus,

E”/E=((aý+bý)^«/(aý+bý)^«).(exp(j•)/exp(-j•))
=exp(j2•)

So the 2• is the phase difference between the incident and reflected wave. So we can also write,

tan(•)=((siný(i)-(n’/n)ý)^«)/cos(i)

(ii) Where E is parallel to the plane of incidence,

E”/E=tan(i-r)/tan(i+r)
=(sin(i)cos(i)-cos(r)sin(r))/(sin(i)cos(i)+cos(r)sin(r))

Substitute for cos(r) (=j((...)^«)) an use Snell's law

=((n'/n)ý.cos(i)+j(siný(i)-(n'/n)ý)^«)/((n'/n)ý.cos(i)-j(siný(i)-(n'/n)ý)^«)

=(a+jb)/(a-jb)

tan(•)=(siný(i)-(n'/n)ý)^«/((n'/n)ý.cos(i))

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For phase changes on reflection (where $i>i_c$), what is the difference in phase change between the two components (E | | and E Á) ?

tan($•_{||}$-$•_{Á}$)=cos(i.(siný(i)-(n'/n)ý)^«)

$•_{||}$-$•_{Á}$ is the difference in phase change and the cosine term is equivalent to **siný(i)**. There are two angles of incidence for which $•_{||}$-$•_{Á}$=pi/4 (45 deg). e.g. for glass to air i=48 deg 47 minutes or 54 deg. 37'.

See fig 2

<u>Incident</u>: Plane of polarased beam (normal incidence). Plane of polarisation 45 deg. to plane of incidence. Equivalent to SHM's, equal in amplitude and at right angles and in phase.

<u>First Reflection</u> Phase difewrece of pi/2 introduced. Equal amplitudes which means it is elliptically polarised.

<u>Second Reflection</u> Another pi/4 phase change is introduced two SHM's at right angles, phase differenmce of pi/2 and equal amplitude. This brings about circular polaraisation.

<u>Frustrated Internal Reflection</u>

Consider a beam passing from a dense medium to a rare medium. In our analysis so far we have assumed a 100% reflection of the E-field. However this is not physically sensible, there must be some finite (albeit tiny) distance over which the E-field attenuates to zero in the medium in which it emerges in. Recall the equation from much earlier,

$u=u_0.\exp(j(\omega t-kx))$

We now write the refracted wave in the same form of the above equation,

$E=E_o.exp[jw(t-((x.sin(r)+z.cos(r))/v'))]$

v' is the velocity of the wave in the new (rare) medium. We can also expand the cosine function,

cos(r)=j.((n'/n)ý.siný(i)-1)^«

Therefore we write the refracted wave as,

$E=E_o$.exp[jw.(t-(x.sin(r)/v'))].exp[-w.((n/n')ý.siný(i)-1)^«.(z/v')]

The decay distance in systems like this is usually over several atomic distances. **Hall Experiment - bean splitter !!! write up !**

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Reflections at dielectric to metal boundaries

Recall the expression for a TEM,

$u=u_o.exp(jwt-yz)=u_o.exp(-az).exp(j(wt$-áz))

We now rewrite this as,

$u=u_o.exp(jwt-n_c.kz)=u_o.exp(-kn_cz).exp(j(wt-kz))$

Where,

k=2pi/lambda, the wave number
$n_c=n_i-jn_r$, this is the complex refractive index. n_i is the attenuation and n_r is the phase velocity.

Fig One

For a good conductor, a=á=(æ•w/2)', likewise $n_i=n_r$=(•/2$E_o w$)' (=a/k).

We now write the reflection coefficient (at normal incidence) as,

R=((n'-n)/(n+n'))ý
=[((n-n_r')+jn_i')/((n+n_r')-jn_i')]ý

Hence we can write the reflection coefficient,

R=1-2n(2wE_o/•)'

Note, the proof for the above statemenets are not within the scope of this course!

Classical theoryof dispersion

So far we have only attributed dispersion to conducting media. However, we know from experiance that dispersion also occurs in dielectric materials. For example, when white light is shone through a triangular glass prism, the wavelength components are split up into a spectrum. For visible light, refractive index decreases with wavelength.

Fig Two

In the graph of Fig. 2, the regions where n decreases with wavevlength is known as **normal dispersion**The regions where n increases with wavelength are known as **anomalous dispersion**
If we now turn our attention to the atoms at the boundary atoms of a material, the **bound electronson atoms** The refractive index curve is imitative of a resonance curve in an oscillatory system. The peaks of the graph

$D=E_o\underline{E}+\underline{P}$ (where P is the bulk polaraization)
$\underline{P}=X.\underline{E}$ (X is the susceptibility)

$K=E/E_o$, this the the dielectric constant.

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<u>Bulk Quantities</u>

$\underline{D}=E_o\underline{E}+\underline{P}$
$\underline{P}=\underline{E}(E-E_o)$
$=\mathrm{E}E_o(K-1)$
$X=\underline{P}/\underline{E}$ - Bulk susceptibility.

All these represent the bulk properties of the material, we now wish to make a molecular analysis (which applies to non-polar dielectrics, i.e. there is no polarization until an external field is applied, furthermore, there is no permanent dipole moment.) of such a system. We can analgously write,

$\underline{p}=a\underline{E}_L$

Where a is the molecular susceptibility, $\underline{E}_L$ is the **local field,** which is due the effect of polarization (from the applied field) and,

$\underline{P}=N\underline{p}$

Where N is the number of molecules.
The local field was analyised by Lorentz and the result obtained is shown below,

$\underline{E}_L=\underline{E}+(\underline{P}/(3\varepsilon_0))$

Now, assume the wavelength of the field (wave) is much greater than the inter molecular separation. Consider the equation of motion,

$mx''+m\gamma x'+m\omega_0^2x=E_L.e$

(ω_0 is the **resonant term**)

We already know that,

$\underline{P}=Ne\underline{x}$

So substituting into the equation of motion,

$P''+\gamma P'+\omega_0^2P=(Ne^2/m).E_L$
$=(Ne^2/m).(E+(P/(3\varepsilon_0)))$

$\underline{P}$ has the same frequency as incident wave but not neccessarily in phase. It is clear now that all of this is analogous to Forced, Damped, SHM.

Consider the incident field,

$E=E_0.\exp(j\omega t)$

Therefore, the bulk polarization can be expressed as follows,

$\underline{P}=\chi\underline{E}$
$=\chi.E_0.\exp(j\omega t)$

$\underline{P}'=j\omega\chi\underline{E}$

$\underline{P}''=-\omega^2\chi\underline{E}$

Now substitute into the equaiton of motion,

$-\omega^2\chi E+\gamma j\omega\chi E+\omega_0^2\chi E$
$=(Ne^2/m).(E+(\chi E/3\varepsilon_0))$

$-\omega^2+\omega_0^2+j\gamma\omega=(Ne^2/m\chi)+(Ne^2/3\varepsilon_0 m)$

Therefore,

$(mX/Ne^2)=1/(w_o^2-w^2+jwy-(Ne^2/3E_om))$

The final term in the denominator of the right hand side of the above equation is the Lorentz local field correction.

$X=P/E=(Ne^2/m).(1/(w_o^2-w^2+jwy-(Ne^2/3E_om)))$

K_e, the relative permattivity can also be expressed as,

$K_e-1=X/E_o=(Ne^2/E_om).(1/(w_o^2-w^2+jwy-(Ne^2/3E_om)))$

Albeit a misnomer, this is also the **dielectric constant** of the material. An additional relation that should be noted is,

$K_e-1=n_c^2-1$

n_c has two components, n_r the phase velocity term and n_i the damping term.

Now let us now write,

$n_c^2-1=(Ne^2/m).(1/(w_o^2-w^2+jwy-(Ne^2/3E_om)))$

We now write (by adding 3 to each side)

$n_c^2+2=3.(w_o^2-w^2+jwy)/(w_o^2-w^2+jwy-(Ne^2/3E_om))$

Now make the step,

$(n_c^2-1)/(n_c^2+2)=(Ne^2/3mE_o).(1/(w_o^2-w^2+jwy-(Ne^2/3E_om)))$

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If we now assume that our medium is a low pressure gas, and hence from our experience of such media, we can say that n_c^2+2 is approximately 3. However this idea looses the Lorentz local field property. If this is the case, we can rewrite the above as,

$(n_c^2-1)=(Ne^2/mE_o).(1/(w_o^2-w^2+jwy)$

If we now make the assumption that $w=w_o$ (approx.) we can write,

$w_o^2-w^2=(w_o+w)(w_o-w)$
$=2w(w_o-w)$

So that we can now write,

$(n_c^2-1)=(Ne^2/mE_o).(1/(2w(w_o-w)+jwy)$

We now hence write,

n_c=[1+{(Neý/mE_o).(1/(2w(w_o-w)+jwy)}]

Also recal that the complex index is written as,

n_c=n_r-jn_i

and using the binomial theorem we can write the approximate relation,

n_c=1+((Neý/2mE_o).(1/(2w(w_o-w)+jwy))

=1+((Neý/2mE_o).((2(w_o-w)-jy)/(4(w_o-w)ý+yý)))

So the components of n_c are,

n_r=1+(Neý/2mwE_o).((2(w_o-w)/(4(w_o-w)ý+yý))

n_i=|(Neý/2mwE_o).(y/(4(w_o-w)ý+yý))|

For more dense media however, we obtain the result,

n_cý-1=(Neý/mE_o).(1/(w_oý-wý+jyw-(Neý/3E_om)))

If we look at two of the terms in the denominator, we make the combination

w_oý-(Neý/3E_om)

and make the quantity of an effective frequency,

(w')ý=w_oý-(Neý/3E_om)

Which shows how much and in what way the Lorentz local field effect affects the medium.

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Scattering

We are concerned for the moment with the efficiency of the scattering mechanism about the scattering centre. We assume for now that the total power scattered (or radiated) by a dipole is,

W=n(pi.I_oý/3).(l/lambda)ý   (watts)

Where n is the intrinsic impedance of the medium, I_o is the current amplitude, l is the length of current dipole and lambda is the wavelength of the incident wavelength. Translating this now into the form of an oscillatory dipole. Since we conisder there to be an oscillatory current due to oscillatory charge carriers, we can write,

$I=I_o.\exp(j\omega t)$
$I=dq/dt=q_o.j\omega.\exp(j\omega t)$ (3)

We also know that the dipole moment is,

p=lq

So the oscillatory dipole moment is,

$=l.q_o.\exp(j\omega t)$
$=p_o.\exp(j\omega t)$ (4)

$lI=lI_o.\exp(j\omega t)=lq_o j\omega \exp(j\omega t)$

Therefore,

|lýI_oý|=|ωýp_oý|

Hence we can rewrite the power scattered as,

W=(pi.n_o.ωý.p_oý.ωý)/(3cý.4.piý)

For free space,

cý=(ωý/4.piý).lambdaý

W=(n_o.ω^{4}.p_oý)/(12.pi.cý)

We now look at N, the mean incident power,

N=«(Eý/n_o)

It can also be shown that,

$\underline{p}_o=a\underline{E}_o$

Where a is the susceptibility. Therefore,

N=(«p_oý)/(aý.n_o)

If we now look at the ratio **W/N**, we write this as,

•=W/N (mý)

• is called the **scattering cross section**. If we now expand the expression for •,

•=W/N=((n_oýω^{4})/(6.pi.cý)).aý

We can see that • is fequency dependant, however we must see how a

behaves with varying frequency. Recall the expression,

P/E=X=(Neý/m).[1/(ω_0ý-ωý+jωy-(Neý/(3mE_0)))]

From the above, assume small damping (y=0) and that our medium is a weak non-polar gas, so there is no Lorentz effect, so we can write,

X=P/E=(Neý/m)(1/(ω_0ý-ωý))

Note that we write P=N.p and P=XE for bulk quantities and p=aE for atomic quantities. Which leads us to stating a=X/N. So we can write,

a=(eý/m).(1/(ω_0ý-ωý))

So do we have • as a function of frequency (or wavelength) ?

(a) $\omega >> \omega_0$ (X-Rays), this gives us,
aý=((eý/mωý))ý, therefore, •=(ýω^4/6picý).(e/mýω)=constant=•$_0$ So the scattering is independant of frequency, this Thomson Scattering
•$_0$=6.65 10^{-21} mý

(b) $\omega << \omega_0$ (Visible)
aý=(eý/mω_0)ý, so, •=•$_0$.(ω/ω_0)^4. We obtain this from substituting into the expression from •. This is known as Rayleigh Scattering i.e. • is proportional to 1/(lambda^4)

Radiation from Dipoles and short current elements

Appendix

Consider a quantity called, the, magnetic vector potential A. Similarly to the H=-Grad (analogous to the Electric vector E=-Grad V). Where A is found as,

A=Integ () ((æJ)/(4pi.r)).dr

Also recall that

delý.V=ç/E
delý.A=æJ

We also have the relation,

B=del A

Recall the relaion,

delýu-æã(Ðu/Ðt)-æE(Ðýu/Ðtý)=0

Where u can be replaced by H,E,V or A.

That equation of field has no term for a source of field. If we assuime •=0 (dielectric), we can write the following,

delýV-æ*E*(ÐýV/Ðtý)=-(1/*E*).ç(x,y,z,t)
For an electric field

delýA-æ*E*(ÐýA/Ðtý)=-æ.J(x,y,z,t)
For a magetic field

These gives the sources of E and H. The general form is,

delýPsi-æ*E*.(ÐýPsi/Ðtý)=-G(x,y,z,t)   (1)

If Psi represents A, G is the current density. If Psi represents V, G is the charge density.

Solution of equation 1 to find V (or A) from a given distrobution of ç (or J).
If Psi is sinusoidal (and assuming G is the same),

G=g.exp(j*w*t)
Psi=psi.exp(j*w*t)

If we consider and arbitrary shaped source (and volume V) and we measure the potential (Psi$_p$)at a point P a distance r from the source. We have the equation,

Psi$_p$=Integ (over V) (([G]/(4pi.r)).d*T*)
=Integ (over V) (((1/4pi.r).g.exp(j*w*(t-r/v))).d*T*)

v is the velocity of the wave from source to observer at P. This also assumes that there is a spherical wave front. Hence we can write particular solutions,

V$_p$=Integ (over V) (([ç]/(4pi.r.*E*)).d*T*)

A$_p$=Integ (over V) ((æ[J]/(4pi.r)).d*T*)

These are called the **retarted potentials**

If we now look at a real example, e.g. short current element.

Fig one

From fig. One,

A$_p$=(æI$_o$l/4pi.r).exp(j(*w*r-kr))

If we want the field strength,

B=del A

=æH

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If we want to find the power of the field, using A, we know that,

H=(del A)/æ

The further evaluation of this all now depends on which coordinate system we use. Here we shall use a kind of cylindrical coordinate.

For A | | to l and | | to z-axis.
A is a function of r only (theta and phi do not appear).

A=(æ$_o$.I$_o$.l/4pi.r).exp(j(ωt-kr))
fig 1

From Fig 1,

h_1=1 U_1=ç $A_ç$(=0)
h_2=ç U_2= A
h_3=1 U_3=Z A_z(<>0)

We further write,

B=del A=(1/ç).ç{(ÐA_z/Ð)-(Ð(çA $_z$/Ðç)-
(Ð$A_ç$/ÐZ)}+Z{(Ð(çA $_ç$)/Ð)}]

(The underlined terms represent unit vectors.

Thankfully, most of this reduces to zero!

$A_ç$=A
and
A_z is not a function of

Leaving us finally only with one non-zero component,
i.e.,

$B_ç$=0
B_z=0
B

B $_z$/Ðç)=-ÐA_z/Ðç

We expand this to,
(recall also, rý=çý+zý and ç=r.sin(theta)

ÐA_z/Ðç=-Ð/Ðç{(exp(jk(çý+Zý)·)/((çý+Zý)·)}.(æ$_o$.I$_o$.exp(jωt).l/4pi)

B $_{\circ}$.I$_{\circ}$.exp(jωt).l/4pi)

Likewise,

H +(jk.exp(-jkr)/r)}.sin(theta).(I$_{\circ}$.exp(jωt).l/4pi)

If we look close to the source, where r is very small, we can rerwrite the above,

H $_{\circ}$.exp(jωt).l/4pi)

This is a form of the **Biot Savart Law** For large values of r we have,

H kr)/r).sin(theta).(I$_{\circ}$.exp(jωt).l/4pi)
H $_{\circ}$l/4pi).exp(j(ωt-kr))

We can see that at near distances we simply have an oscillating field (called the **induction** field). At large distances however, we can readily see that the H field behaves like a progressive wave (it is called the **Radiation** field).

The corresponding electric field comes from the theta component. We know that,

CURL H = ÐD/Ðt =*E*(ÐE/Ðt)

Hence it (does!) follow that,

E_{theta}=(jωæ/r).sin(theta).(I$_{\circ}$l/4pi).exp(j(ωt-kr))

fig 2

If we take the ratio of H $_{theta}$, since there is j in both quantities so the complex component cancels out, H and E are in phase with each other.

E_{theta}/H *E*)=*n* intrinsic impedance
((æ$_{\circ}$/*E*$_{\circ}$)=*n*$_{\circ}$=120.pi ohms)

We can now also evaluate the radiated power (in reference to the poynting vector),

P=«(E$_{\circ}$ H)
=«(E_{theta} H
=«(E_{theta}ý/*n*=«(H *n*)
=«*n*[kI$_{\circ}$l.sin(theta)/4pi.r]ý (Watts/mý)
(fig 3)

The total power is obtai.ned from integration over the surface of a sphere around the (anisotropic) radiation pattern. The total power can be expressed as,

W=Integ (0,pi) {P_{av}.2pi.rý.sin(theta).dtheta}
(fig 4)

W=(ký.I_oý.n.lý/16(piý)) . Integ (0,pi) (sinü(theta).dtheta)
W=(1/3).n.pi.I_oý.(l.lambda)ý (watts)

Aerial gain

Aerial gain= (P_{av} (theta=90))/(Total Power)
=(P_{av}/(W/4pi.rý))

Where the denominator gives the distrobution of total power. An aerial gain of 1.5 is characteristic of dipole radiaton.

References

Jenkins and White, fundamentals of geometery and optics